The Baire Category Theorems

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In trying to learn about the Baire category theorems (BCTs), I could not find any self-contained, proofs of the BCTs in full generality. It is for this reason I have written the following paper. For a more in depth introduction to the BCTs as well as general topology, I suggest *General Topology* by John L. Kelly.

1 Introduction

The Baire Category Theorems consists of two conditions sufficient for a topological space to be Baire (See Definition 3). They are an important result in general topology used in a number of results in other branches of matheamtics, particularly functional analysis. For a survey of results relying on the BCTs, I suggest *Applications of the Baire Category Theorem* by Sara H. Jones.

In any topological space, the intersection of finitly many open dense sets is dense by Lemma 1.

Lemma 1. Finite intersections of dense open sets are dense.

Proof. It follows from the definition that a set is dense iff it intersect every non-empty open set. Let U and U' be dense open sets, and let V be an arbitrary non-empty open set. Then, $V \cap U \neq \varnothing$, so $(V \cap U) \cap U' \neq \varnothing$, so $V \cap (U \cap U') \neq \varnothing$. Since V is arbitrary and non-empty, $U \cap U'$ is dense. ∎

However, it is not in general true that infinite countable intersections of open dense sets are dense. Consider the following example.

Example 2. Let $\mathbb{Q} = \{q_1, q_2, \cdots\}$ be a topological space under the metric topology, and consider the sets

$$U_n = \mathbb{Q} \setminus \{q_n\}.$$

$\{q_n\}$ is closed, so U_n is open. U_n is dense in $\mathbb{Q}$ since for any $p \in \mathbb{Q}$, either $p \in U_n$, or $p = q_n$ in which case the sequence $(p - \frac{1}{n})_{n=1}^{\infty}$ belongs to U_n and converges to p. However,

$$\bigcap_{n=1}^{\infty} U_n$$

is clearly empty, and therefore not dense in $\mathbb{Q}$.

It may be desirable for infinite countable intersections of open dense sets to be dense, hence why we study the notion of a Baire space.

Definition 3. A topological space is **Baire** if countable intersections of dense open sets are dense.

The Baire Category Theorem in full generallity follows.

Theorem 4 (Baire Category Theorems)**.**

(a) Every complete pseudometric space is a Baire space.

(b) Every locally compact regular space is a Baire space.

Note that neither of these conditions imply the other, nor are either necessary for a topological space to be Baire. For example, the lower-limit topology on $\mathbb{R}$ lends a topological space which is Baire while neither pseudometric nor locally compact. For more information, see Wikipedia: Lower Limit Topology.

2 The Baire Category Theorem Part (a)

Definition 5. Let X be a topological space. A collection $\mathcal{B}$ of neighborhoods of of x is a **local base** if for every open set U containing x, there exists some $B \in \mathcal{B}$ such that $x \in B \subset U$.

The existence of a local base possible satisfying some additional property is useful since it garuentees the existence of open sets with that additional property "between" any point and any neighborhood of that point.

Remark. In the language of filters, a local base is a filter basis for the neighborhood filter $\mathcal{N}(x)$. This definition is one example of where an understanding of filters may prove useful. In order to prove part (a) of the Baire category theorem, we construct a Cauchy sequence which converges to a point which all our open dense sets intersect. Sequences behave poorly in non-metric or non-Housdorff spaces since, for example, in such spaces we cannot recover the topology of our space from convergant sequences. However, filters provide a more general, and better behaved framework for deeling with analogous constructions. The key to both parts of this proof is constructing a descending chain of non-empty subsets

$$U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots$$

and showing that this sequence "converges" in a sense made precise by filters. Along these lines, filters can be used to unify and the proof of part (a) and part (b) of the Baire category theorem. Unfortunately, this is out of the scope of this paper. For a brief introduction to nets and filters, I once again suggest *General Topology* by John L. Kelly.

Lemma 6. In a pseudometric space, every point has a local base of closed neighborhoods. Furthermore, for any point x, neighborhood U, and real number $\varepsilon > 0$, there exists a closed set $\mathrm{Cl}(V)$ such that $\mathrm{Cl}(V) \subset U$ of diameter at most ε.

Proof. The psuedometric topology on X is given by the basis

$$\{B_\varepsilon(x) \mid x \in X, \varepsilon \in \mathbb{R}_{>0}\},$$

so there exists some ball $B_\varepsilon(y)$ containing x and contained in $\mathrm{Int}(U) \subseteq U$. We have $\mathrm{Cl}(B_{\frac{1}{2}(\varepsilon - d(x,y))}(x)) \subset B_\varepsilon(y) \subseteq U$. That is, x has a local base given by closed sets, and we may take these closed sets to be arbitrarally small.

■

Lemma 7. Let $U = (U_i)_{i=1}^\infty$ be a countable collection of nonempty sets satisfying the descending chain

$$U_1 \supseteq U_2 \supseteq \cdots .$$

Then, U has the finite intersection property.

Proof. For any positive integer n, $U_1 \supseteq \cdots \supseteq U_n \neq \varnothing$, so $U_1 \cap \cdots \cap U_n \subseteq U_n$. For any finite collection $\{U_{i_1}, \cdots, U_{i_m}\}$, we have

$$U_1 \cap \cdots \cap U_{\max(i_1,\cdots,i_m)} \subseteq U_{i_1} \cap \cdots \cap U_{i_m} \subseteq U_{i_m}.$$

U_n is nonempty, so any collection $\{U_{i_1}, \cdots, U_{i_m}\}$ has the finite intersection property. ■

We are not ready to prove part (a) of the Baire category theorems. The following is based on the proof given in *General Topology* by John L. Kelly.

Proof of Theorem 4 part (a). Let $U_1, U_2, \cdots$ be a countable collection of dense open sets. We will show any arbitrary non-empty open set W has a point x in common with U_n. Since a set is dense iff every nonempty open subset intersects it, we will conclude that the intersection of all U_n is dense.

Both W and U_1 are open, so $W \cap U_1$ is open. Since U_1 is dense, $W \cap U_1$ is nonempty, so we may take an arbitrary point in $W \cap U_1$ and apply Lemma 6 to get a dense open set V_1 such that $\mathrm{Cl}(V_1)$ is a closed subset of $W \cap U_1$ of, withouut loss of generality, diameter less than 1. Similarly, for $n > 1$, let V_n be an open set such that $\mathrm{Cl}(V_n)$ is a closed subset of $V_{n-1} \cap U_n$ of diameter less than $\frac{1}{n}$. It follows from Lemma 7 that $\{\mathrm{Cl}(V_1), \mathrm{Cl}(V_2), \cdots\}$ satisfies the finite intersection property. Since $\mathrm{Cl}(V_i)$ is nonempty and of diameter less than or equal to $\frac{1}{i}$, by choosing some x_i from each $\mathrm{Cl}(V_i)$, we obtain a Cauchy sequence $(x_j)_{j=1}^{\infty}$. We are working in a complete space, so $(x_j)_{j=1}^{\infty}$ converges to some point x. Every $\mathrm{Cl}(V_i)$ contains a tail of $(x_j)_{j=1}^{\infty}$, so $x \in \mathrm{Cl}(V_i)$. Thus, $x \in W$ and $x \in U_i$, so $W \cap \bigcap_{i=1}^{\infty} U_i$ is nonempty. W is an arbitrary open set, so $U = \bigcap_{i=1}^{\infty} U_i$ is dense. ■

Example 8. From part (a) of the Baire category theorem, the following are Baire spaces.

- Both $\mathbb{R}$ and $\mathbb{C}$
- For some set X, the space $\mathcal{F}(X)$ consisting of real-valued functions $X \to \mathbb{R}_{\geq 0}$ under the topology induced by the pseudometric

$$d : \mathcal{F}(X) \times \mathcal{F}(X) : f, g \mapsto |f(x_0) - g(x_0)|.$$

3 Compact Spaces

We list several standard, relevant charecterizations of compact and locally compact spaces.

Lemma 9. Compact subsets of a Housdorff space are closed.

Proof. See nCatLab: compact subspaces of Hausdorff spaces are closed. ■

Lemma 10. Closed subspaces of compact spaces are compact.

Proof. See nCatLab: closed subspaces of compact spaces are compact . ■

Lemma 11. Continuous images of compact spaces are compact.

Proof. See nCatLab: continuous images of compact spaces are compact. ■

Lemma 12. Subsets are closed in a closed subspace iff they are closed in the ambient space.

Proof. See nCatLab: subsets are closed in a closed subspace precisely if they are closed in the ambient space. ■

Lemma 13. Topological space is compact iff any collection of closed sets with the finite intersection property has a non-empty intersection.

Proof. These are equivalent statements in opposite categories. In particular, compactness means that for any fammily $\mathcal{O}$ of open sets,

$$\bigcup_{O \in \mathcal{O}} O = X \quad \Longrightarrow \quad \exists \text{ finite } \mathcal{A} \subset \mathcal{O} : \bigcup_{O \in \mathcal{A}} O = X.$$

By taking the contrapositive, we have

$$\bigcup_{O \in \mathcal{O}} O \neq X \implies \forall \text{ finite } \mathcal{A} \subset \mathcal{O} : \bigcup_{O \in \mathcal{A}} O \neq X.$$

By "translating" to statements in the opposite category,

$$\bigcap_{C \in \mathcal{C}} C \neq \varnothing \impliedby \forall \text{ finite } \mathcal{B} \subset \mathcal{C} : \bigcap_{C \in \mathcal{B}} C \neq \varnothing.$$

The ladder is precisely the second property. ■

We now charecterize regular and locally compact Housdorff spaces in terms of local bases.

Lemma 14. A topological space is regular iff every point has a local base of closed neighborhoods.

Proof. ($\Rightarrow$) Choose any point $x \in X$ and neighborhood U of x. X is regular, so there exist disjoint open sets A and B such that $x \in A$ and $X \backslash U \subset B$. $X \backslash B$ is a closed neighborhood of x satisfying $X \backslash B \subset U$.

($\Leftarrow$) Choose any point x and closed set U not containing x. X/U is an open neighborhood of x, so it contains a closed neighborhood $V \subseteq X/U$ of x. Then, $X/V \supseteq U$ is an open neighborhood of U not containing x, and $\text{Int}(V)$ is an open neighborhood of x disjoint from X/V. Hence, X is regular. ■

Lemma 15. If a topological space is locally compact regular, then every point has a local base of closed compact neighborhoods.

Proof. For any $x \in X$, let U be an open neighborhood of x, and from Lemma 14 there exists a closed subset A of U containing x. Since X is locally compact, there exists a compact neighborhood B of x. From Lemma 9, B is closed. Hence, $A \cap B$ is closed. Since $A \cap B \subset B$, from Lemma 10, $A \cap B$ is compact. $x \in A \cap B \subset A \subset U$, so $A \cap B$ is a compact neighborhood of x contained in U. By Lemma 9, $A \cap B$ is closed. ■

Lemma 16. A locally compact Housdorff space is regular.

Proof. The following proof is based on nCatLab: locally compact topological space.

We will show that in locally comact Housdorff spaces, every point x has a local basis of closed compact neighborhoods. It follows from Lemma 14 that such spaces are regular.

For any point x, and any open neighborhood U_x of x, since X is locally compact, there exists a compact neighborhood $K_x \subseteq U_x$ of x. We have the inclusions

$$\{x\} \subseteq \text{Int}(K_x) \subseteq K_x \subseteq U_x \subseteq X.$$

By Lemma 9, K_x is closed. Hence, $\text{Cl}(\text{Int}(K_x)) \subseteq K_x$. From Lemma 12, $\text{Cl}(\text{Int}(K_x))$ is a closed subset of the compact subspace K_x. From Lemma 10, $\text{Cl}(\text{Int}(K_x))$ is compact in K_x. From Lemma 11, $\text{Cl}(\text{Int}(K_x))$ is a compact subspace of X. We have the inclusion

$$\{x\} \subseteq \text{Cl}(\text{Int}(K_x)) \subseteq U_x \subseteq X.$$

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4 The Baire Category Theorem Part (b)

Note the similarity between parts (a) and (b) of the proof of Theorem 4.

Proof of Theorem 4 part (b). Let $U_1, U_2, \cdots$ be a countable collection of dense open sets. We will show any arbitrary non-empty open set W has a point x in common with U_n. Since a set is dense iff every nonempty open subset intersects it, we will conclude that the intersection of all U_n is dense.

Both W and U_1 are open, so $W \cap U_1$ is open. Since U_1 is dense, $W \cap U_1$ is nonempty, so we may take an arbitrary point in $W \cap U_1$ and apply Lemma 15 to get a dense open set V_1 such that $\mathrm{Cl}(V_1)$ is a closed compact subset of $W \cap U_1$. Similarly, for $n > 1$, let V_n be an open set such that $\mathrm{Cl}(V_n)$ is a closed compact subset of $V_{n-1} \cap U_n$. It follows from Lemma 7 that $\{\mathrm{Cl}(V_1), \mathrm{Cl}(V_2), \cdots\}$ satisfies the finite intersection property. By Lemma 13, $\bigcap_{i=1}^{\infty} \mathrm{Cl}(V_i)$ is nonempty. That is, there exists some point x such that $x \in \mathrm{Cl}(V_i)$ for all i. Thus, $x \in W$ and $x \in U_i$, so $W \cap \bigcap_{i=1}^{\infty} U_i$ is nonempty. W is an arbitrary open set, so $U = \bigcap_{i=1}^{\infty} U_i$ is dense. ■

Example 17. From part (b) of the Baire category theorem, the following are Baire spaces.

- Any Hausdorff topological manifold is locally compact since there is a neighborhood of every point which is homeomorphic to the locally compact space $\mathbb{R}^n$, and regular by Lemma 16.