The Utility of Completions

Brandon Feder
September 23rd, 2023

For the remainder of this document, we will assume all rings are commutative with unity. Let k be an algebraically closed field, let V be a (closed) subvariety, let $R = k[V]$ be the coordinate ring of V, let $p \in V$ be a point, and let $\mathfrak{m}$ be the coordinate ring of p.

The localization of R at $S = R \backslash \mathfrak{m}$ is usually denoted $R_\mathfrak{m}$. $R_\mathfrak{m}$ is a local ring with the unique maximal ideal $\mathfrak{m}R_\mathfrak{m}$. The completion of R at $\mathfrak{m}$ is defined to be

$$\hat{R}_\mathfrak{m} := \varprojlim (R/\mathfrak{m}^n).$$

Intuitively, if a function f is defined in a Zariski neighborhood U of V, and it does not vanish on V, then after removing $V(f)$ from U we still have a neighbourhood of V, but for which f is invertable. Similarly, by shrinking the support of f around V, eventually f will vanish or become a unit. In this sense, the localization of R at a maximal ideal $\mathfrak{m}$ in a sense is the algebraic analog of the ring of germs of p. However, the completion of $R_\mathfrak{m}$ represents the properties of the variety in "far smaller" neighborhoods.

Consider the cubic nodal plain curve $y^2 - x^2(x+1)$ and the curve $y^2 - x^2$ which is just a pair of lines. Under the standard metric topology, on sufficiently small intervals these two curves look identical. However, their local rings at $\mathfrak{m} = (x, y)$ act differently from one another.

Since the nodal plain curve is irreducible, its coordinate ring $k[x,y]/(y^2 - x - 1)$ is a domain, and it follows immediately that the localization of this curve at $\mathfrak{m}$ is also a domain, and therefore irreducible. This is not the case for $y^2 - x^2$.

On the other hand, in $\hat{k}[x,y]_{(x,y)}/(y^2 - x - 1)$, $1 + x$ has a square root given by its Taylor series

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{32}x^2 + \cdots,$$

so $y^2 - x^2(1+x)$ is reducible.

Another example: Consider the parabola $y^2 - x - 1$ and the line y which have coordinate rings $k[x,y]/(y^2 - x - 1)$ and $k[x]$ respectively. The projection $\pi : y \mapsto 0$ gives a two-to-one map from the parabola into the line. The map

$$\pi^\# : k[x] \to k[x,y]/(y^2 - x - 1) : x \mapsto x$$

can be thought of as the inclusion of the coordinate ring of the line into the coordinate ring of the parabola. The derivative of π at $x = 0$ is nonzero, so by the inverse function theorem, π has a local inverse at $(0, -1)$. This inverse is given by the analytic function

$$\sigma : k \to k : x \mapsto \sqrt{x+1}$$

which is not a polynomial. However, this function is represented by the aforementioned Tayor series, and so at the scale of the completion of the two coordinate rings,

$$\sigma^\# : \hat{k}[x,y]/(y^2 - x - 1) \to \hat{k}[x] : x \mapsto x, y \mapsto -\sqrt{x+1}.$$

For more about completions and localizations, I highly suggest *Commutative Algebra with a View Toward Algebraic Geometry* by David Eisenbud.