The Finitary Giry Functor

Brandon Feder
July 11th, 2024

Given a set X, define $\mathcal{P}X$ as the set whose elements are finitely supported probability measures over X, i.e. functions $p : X \to [0, 1]$ such that $\sum_{x \in X} p(x) = 1$ and $p(x) \neq 0$ for only finitely many x. Given a function $f : X \to Y$, let $\mathcal{P}f : \mathcal{P}X \to \mathcal{P}Y$ take $p \in \mathcal{P}X$ to the function

$$y \mapsto \sum_{x \in f^{-1}(y)} p(x). \tag{1}$$

Lemma. $\mathcal{P}f(p)$ is a finitely supported probability measure over Y.

Proof. $\mathcal{P}f(p)$ is a probability measure over Y since

$$\sum_{y \in Y} \mathcal{P}f(p)(y) = \sum_{y \in Y} \sum_{x \in f^{-1}(y)} p(x) = \sum_{x \in f^{-1}(Y)} p(x) = \sum_{x \in X} p(x) = 1.$$

Since every element $x \in X$ appears in the sum in Eq. (1) for a single y, and because there are finitely many x for which $p(x) \neq 0$, it must be that there are only finitely many y such that $\mathcal{P}f(p)(y) \neq 0$. ■

Theorem. $\mathcal{P}$ is an endofunctor on **Set**.

Proof. Clearly $\mathcal{P}$ associates to each object (resp. morphism) of **Set** another object (resp. morphism) of **Set**.

Fix an object X. By construction, $\mathcal{P}\mathrm{id}_X$ takes an element $p \in \mathcal{P}X$ to the function

$$y \mapsto \sum_{x \in \mathrm{id}_X^{-1}(y)} p(x),$$

but $\mathrm{id}_X^{-1}(y) = \{y\}$, so p is mapped to $y \mapsto p(y)$ which is the identity of $\mathcal{P}X$.

For any morphisms $f : X \to Y$ and $g : Y \to Z$, we have $\mathcal{P}(g \circ f)$ takes $p \in \mathcal{P}X$ to the map

$$\begin{aligned} & y \mapsto \sum_{x \in (g \circ f)^{-1}(y)} p(x) \\ = \quad & y \mapsto \sum_{x \in (f^{-1} \circ g^{-1})(y)} p(x) \\ = \quad & y \mapsto \sum_{x \in g^{-1}(y)} \sum_{z \in f^{-1}(x)} p(z) \\ = \quad & \mathcal{P}f \circ \mathcal{P}g. \end{aligned}$$

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Remark. We may also make $\mathcal{P}$ into an endofunctor on **Meas** by endowing $\mathcal{P}X$ with the initial σ-algebra of evaluation maps

$$i_A : \mathcal{P}X \to [0, 1] : p \mapsto p(A).$$

Let X be a set and fix some $x \in X$, and let δ_x be the Dirac function defined by

$$\delta_x : X \to \{0, 1\} : a \mapsto \begin{cases} 1 & x = a \\ 0 & x \neq a \end{cases}.$$

Clearly $\delta_x \in \mathcal{P}X$.

Theorem. The family of morphisms

$$\Delta_X : X \to \mathcal{P}X : x \mapsto \delta_x$$

is natural in X. That is, they are the components of a natural transformation $\mathrm{id}_{\mathbf{Set}} \Rightarrow \mathcal{P}$.

Proof. Note that Δ_X is injective. From Eq. (1) we see that for all $y \in Y$, $\mathcal{P}f(\delta_x)(y) = 1$ iff $y = f(x)$. That is, $\mathcal{P}f(\delta_x)(y) = \delta_{f(x)}$. Hence,

$$\Delta_Y \circ f = \mathcal{P}f \circ \Delta_X.$$

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